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SAMPLE PAPER TEST – 01 (2020-21)

SAMPLE ANSWERS

SUBJECT: MATHEMATICS | **MAX. MARKS : 80**

CLASS : XII | **DURATION : 3 HRS**

General Instruction:

1. This question paper contains two **parts A and B**. Each part is compulsory. Part A carries **24** marks and Part B carries **56** marks
2. **Part-A** has Objective Type Questions and **Part -B** has Descriptive Type Questions
3. Both Part A and Part B have choices.

Part – A:

1. It consists of two sections**- I and II**.
2. Section **I** comprises of 16 very short answer type questions.
3. Section **II** contains **2** case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt **any 4 out of 5 MCQs**.

Part – B:

1. It consists of three sections**- III, IV and V.**
2. Section **III** comprises of 10 questions of **2 marks** each**.**
3. Section **IV** comprises of 7 questions of **3 marks** each**.**
4. Section **V** comprises of 3 questions of **5 marks** each**.**

PART - A

SECTION-I

Questions 1 to 16 carry 1 mark each.

1. Show that $f(x) = x^2 + 2$ is not injective.

We have, $f(x) = x^2 + 2$

Let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^2 + 2 = x_2^2 + 2$$

$$\Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$$

$\therefore f(x)$ is not one-one.

2. For any matrix $A = [a_{ij}]$, if c_{ij} denotes its cofactors then find the value of $a_{11}c_{12} + a_{12}c_{22} + a_{13}c_{32}$.
Ans: Zero

3. Find the principal value of $\sec^{-1}(-\sqrt{2}) + \operatorname{cosec}^{-1}(-\sqrt{2})$

$\sec^{-1}(-\sqrt{2}) + \operatorname{cosec}^{-1}(-\sqrt{2})$

$$= \left(\pi - \frac{\pi}{4}\right) + \left(-\frac{\pi}{4}\right) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

4. Check whether the relation R defined on the set A = {1, 2, 3, 4, 5, 6} as R = {(a, b): b= a + 1} is reflexive.

The relation R on set $A = \{1, 2, 3, 4, 5, 6\}$ is defined as

$R = \{(a, b) : b = a + 1\}$

$\therefore R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

Clearly, $(a, a) \notin R$ for any $a \in A$. So, R is not reflexive.

5. If $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ be such that $A^{-1} = kA$, then find the value of k.

We have, $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$

$\therefore \ |A| = 2(-2) - 3\times 5 = -19$

$\therefore \ A^{-1} = \frac{-1}{19}\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix} = \begin{bmatrix}2/19 & 3/19\\ 5/19 & -2/19\end{bmatrix}$

Since, $A^{-1} = kA$

$\Rightarrow \begin{bmatrix}2/19 & 3/19\\ 5/19 & -2/19\end{bmatrix} = \begin{bmatrix}2k & 3k\\ 5k & -2k\end{bmatrix} \Rightarrow \ 2/19 = 2k \ \Rightarrow \ k = \frac{1}{19}$

6. If A is a matrix of order 3 x 3 such that |A| = 5, find |A(adjA)|.

We have, $|A| = 5$ and order of A is 3×3

Now, $|A \,\mathbf{adj}\,(A)| = |A||\,\mathbf{adj}\,A| = |A||A|^{3-1}$

$= |A|^3 = (5)^3 = 125$

7. Find the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\frac{dy}{dx}$.

We have, $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$

Clearly, the degree of differential equation is not defined.

8. Evaluate: $\int \sin^2\frac{x}{2}dx$

$\int \sin^2\frac{x}{2}dx = \frac{1}{2}\int 2\sin^2\frac{x}{2}dx$ [multiply and divide by 2]

$= \frac{1}{2}\int (1-\cos x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos x\, dx$

$= \frac{1}{2}x - \frac{1}{2}\sin x + C$

9. Evaluate: $\int_0^2 e^{[x]}dx$

Let $I = \int_0^2 e^{[x]}dx = \int_0^1 e^{[x]}dx + \int_1^2 e^{[x]}dx$

$= \int_0^1 e^0\, dx + \int_1^2 e^1\, dx$

$= [x]_0^1 + e[x]_1^2 = (1-0) + e(2-1) = e + 1$

10. If A and B are two independent events with P(A) = 3/5 and P(B) = 4/9, then find $P(\overline{A}\cap\overline{B})$.

We know that, if A and B are independent events, then
$\overline{A}$ and $\overline{B}$ are also independent.

$\therefore \ P(\overline{A}\cap\overline{B}) = P(\overline{A})\cdot P(\overline{B}) = (1-P(A))\,(1-P(B))$

$= \left(1-\frac{3}{5}\right)\left(1-\frac{4}{9}\right) = \frac{2}{5}\times\frac{5}{9} = \frac{2}{9}$

11. If E_1 and E_2 are two independent events such that $P(E_1) = 0.35$ and $P(E_1 \cup E_2) = 0.60$, then find $P(E_2)$.

Let $P(E_2) = x$

Then, E_1 and E_2 being independent events, we have

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = 0.35x$$

Now, $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$\Rightarrow 0.60 = 0.35 + x - 0.35x \Rightarrow 0.65x = 0.25$$

$$\therefore x = \frac{0.25}{0.65} = \frac{5}{13}$$

12. If the angle between the vectors $\hat{i}+\hat{k}$ and $\hat{i}-\hat{j}+\alpha\hat{k}$ is $\frac{\pi}{2}$, then find the value of α.

Let $\vec{a} = \hat{i} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \alpha\hat{k}$.

Since, $\vec{a}$ and $\vec{b}$ are perpendicular

$$\therefore \vec{a}\cdot\vec{b} = 0 \Rightarrow (\hat{i} + \hat{k})(\hat{i} - \hat{j} + \alpha\hat{k}) = 0$$

$$\Rightarrow 1 + \alpha = 0 \Rightarrow \alpha = -1$$

13. If $\vec{a} = (\hat{i}+3\hat{j}-2\hat{k})\times(-\hat{i}+3\hat{k})$, then find the value of $|\vec{a}|$.

We have, $\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 0\hat{j} + 3\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9-0)\hat{i} - (3-2)\hat{j} + (0+3)\hat{k}$$

$$= 9\hat{i} - \hat{j} + 3\hat{k}$$

$$\therefore |\vec{a}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{91}$$

14. If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}.\vec{b} = 3$, then find projection of $\vec{b}$ on $\vec{a}$.

Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} = \dfrac{3}{2}$

15. Find the projection of the vector $7\hat{i}+\hat{j}-4\hat{k}$ and $2\hat{i}+6\hat{j}+3\hat{k}$.

We know that, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Here, $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

$$\therefore \vec{a}\cdot\vec{b} = (7\hat{i} + \hat{j} - 4\hat{k})\cdot(2\hat{i} + 6\hat{j} + 3\hat{k}) = 14 + 6 - 12 = 8$$

and $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{4 + 36 + 9} = 7$

Hence, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{8}{7}$.

16. Find the value of λ if the vector $2\hat{i}+\lambda\hat{j}-4\hat{k}$ and $2\hat{i}-\hat{j}+\hat{k}$ are perpendicular.

Since $2\hat{i} + \lambda\hat{j} - 4\hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$ are perpendicular

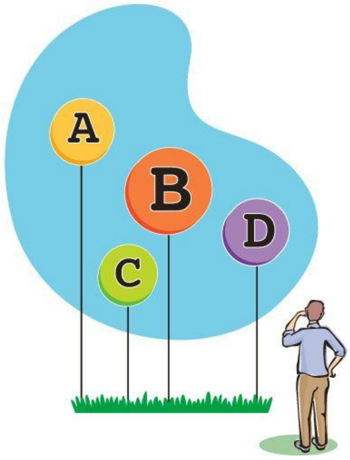
$$\therefore (2\hat{i} + \lambda\hat{j} - 4\hat{k})\cdot(2\hat{i} - \hat{j} + \hat{k}) = 0$$

$$\Rightarrow 4 - \lambda - 4 = 0 \Rightarrow \lambda = 0$$

SECTION-II

Case study-based questions are compulsory. Attempt any four sub parts of each question. Each subpart carries 1 mark

17. In a test, you either guesses or copies or knows the answer to a multiple-choice question with four choice. The probability that you make a guess is $\frac{1}{3}$, you copy the answer is $\frac{1}{6}$. The probability that your answer is correct, given that you guess it, is $\frac{1}{8}$. And also, the probability that you answer is correct, given that you copy it, is $\frac{1}{4}$.



(i) The probability that you know the answer is

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Ans: (c) $\frac{1}{2}$

(ii) The probability that your answer is correct given that you guess it, is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{4}$

Ans: (b) $\frac{1}{8}$

(iii) The probability that your answer is correct given that you know the answer, is

(a) $\frac{1}{7}$ (b) 1 (c) $\frac{1}{9}$ (d) $\frac{1}{10}$

Ans: (b) 1

(iv) The probability that you know the answer given that you correctly answered it, is

(a) $\frac{4}{7}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) None of these

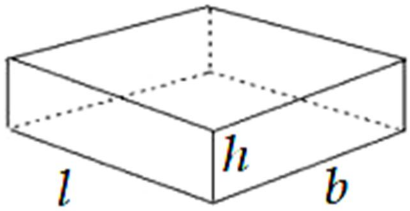
Ans: (c) $\frac{6}{7}$

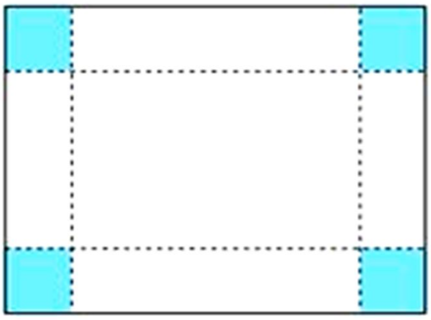
(v) The total probability of correctly answered the question, is

(a) $\frac{7}{12}$ (b) $\frac{11}{12}$ (c) $\frac{5}{12}$ (d) None of these

Ans: (a) $\frac{7}{12}$

18. A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to forms a box.





(i) The length, breadth and height of the box formed, in terms of x are

(a) (24 – 2x, x, 24 – 2x)　　(b) (24 – 2x, 24 – 2x, x)

(c) (x, 24 – 2x, 24 – 2x)　　(d) (x, x, x,)

Ans: (b) (24 – 2x, 24 – 2x, x)

(ii) Volume V of the box expressed in terms of x is

(a) $(24 – 2x)^3$　　(b) x^3　　(c) $x^2(24 – 2x)$　　(d) $x(24 – 2x)^2$

Ans: (d) $x(24 – 2x)^2$

(iii) Maximum value of volume of box is

(a) 256 cm^3　　(b) 512 cm^3　　(c) 1024 cm^3　　(d) 2048 cm^3

Ans: (c) 1024 cm^3

(iv) The value of x when volume is maximum is

(a) 12 cm　　(b) 8 cm　　(c) 4 cm　　(d) 2 cm

Ans: (c) 4 cm

(v) The cost of box, if rate of making the box is Rs. 5 per cm^2, when volume is maximum is

(a) Rs. 2840　　(b) Rs. 3840　　(c) Rs. 2040　　(d) Rs. 3480

Ans: (b) Rs. 3840

PART B
SECTION – III
Questions 19 to 28 carry 2 marks each.

19. Find the values of 'a' so that the function f(x) is defined by $f(x)=\begin{cases}\dfrac{\sin^2 ax}{x^2}, & x\neq 0\\ 1 & , x=0\end{cases}$ may be continuous at x = 0.

The function $f(x)$ will be continuous at $x = 0$, iff

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0}\frac{\sin^2 ax}{x^2} = 1 \quad [\because f(0)=1]$$

$$\Rightarrow a^2 \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)^2 = 1$$

$$\Rightarrow a^2(1)^2 = 1$$

$$\therefore a = \pm 1$$

Thus, $f(x)$ will be continuous at $x = 0$, if $a = \pm 1$.

20. If $\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3$, then find the value of x.

$$\begin{vmatrix} x-2 & -3 \\ 3x & 2x \end{vmatrix} = 3 \Rightarrow 2x(x-2)+9x = 3$$

$$\Rightarrow 2x^2 - 4x + 9x - 3 = 0$$

$$\Rightarrow 2x^2 + 5x - 3 = 0 \Rightarrow x = \frac{1}{2}, -3$$

21. Evaluate: $\sec^{-1}\sqrt{2} + 2\operatorname{cosec}^{-1}(-\sqrt{2})$.

Let $\sec^{-1}\sqrt{2} = \theta_1 \Rightarrow \sec\theta_1 = \sqrt{2}$

$$\Rightarrow \sec\theta_1 = \sec\frac{\pi}{4} \Rightarrow \theta_1 = \sec^{-1}\sqrt{2} = \frac{\pi}{4}\left[\because \frac{\pi}{4} \in [0,\pi] - \left\{\frac{\pi}{2}\right\}\right]$$

and $\operatorname{cosec}^{-1}(-\sqrt{2}) = \theta_2 \Rightarrow \operatorname{cosec}\theta_2 = -\sqrt{2}$

$$\Rightarrow \operatorname{cosec}\theta_2 = -\operatorname{cosec}\frac{\pi}{4} \Rightarrow \operatorname{cosec}\theta_2 = \operatorname{cosec}\left(-\frac{\pi}{4}\right) \Rightarrow \theta_2 = \operatorname{cosec}^{-1}(-\sqrt{2}) = -\pi/4$$

$$\left[\because -\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}\right]$$

$$\therefore \sec^{-1}\sqrt{2} + 2\operatorname{cosec}^{-1}(-\sqrt{2}) = \frac{\pi}{4} + 2\left(-\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

22. Evaluate: $\int \frac{dx}{9x^2+6x+10}$.

Let $I = \int \frac{1}{9x^2+6x+10}dx = \frac{1}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{10}{9}}dx$

$$= \frac{1}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}+\frac{10}{9}}dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+(1)^2}dx = \frac{1}{9}\times\frac{1}{1}\tan^{-1}\left(\frac{x+\frac{1}{3}}{1}\right)+C$$

$$= \frac{1}{9}\tan^{-1}\left(\frac{3x+1}{3}\right)+C$$

23. If $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, $\vec{a} \neq 0$ and $\vec{b} \neq \vec{c}$, show that $\vec{b} = \vec{c} + t\vec{a}$ for some scalar t.

We have, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$,

$$\Rightarrow \vec{a}\times\vec{b} - \vec{a}\times\vec{c} = 0 \Rightarrow \vec{a}\times(\vec{b}-\vec{c}) = 0 \Rightarrow \vec{a} = 0$$

or $\vec{b} - \vec{c} = 0$

or $\vec{a} \,||\, (\vec{b}-\vec{c}) \Rightarrow \vec{a} = 0$

or $\vec{b} = \vec{c}$

or $\vec{a} \,||\, (\vec{b}-\vec{c})$

$\Rightarrow \vec{b} - \vec{c} = t\vec{a}$ for some scalar t. $[\because \vec{a} \neq 0$ and $\vec{b} \neq \vec{c}]$

$$\Rightarrow \quad \vec{b} = \vec{c} + t\vec{a}$$

24. A couple has two children. Find the probability that both children are males, if it is known that atleast one of the children is male.

Let A and B be the events that both children are male and atleast one children is male.

$\therefore \quad P(A) = \frac{1}{4}, \quad P(B) = \frac{3}{4}, \quad P(A \cap B) = \frac{1}{4}$

Now, required probability, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$

25. Evaluate: $\int_0^{\pi/4} \sqrt{1+\sin 2x}\; dx$

Let $I = \int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$. Then,

$$I = \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx = \int_0^{\pi/4} \sqrt{(\sin x + \cos x)^2} dx$$

$$\Rightarrow I = \int_0^{\pi/4} |\cos x + \sin x|\, dx = \int_0^{\pi/4} (\cos x + \sin x) dx = [\sin x - \cos x]_0^{\pi/4}$$

$$\Rightarrow I = \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (0-1) = 1$$

26. If $\sec\left(\frac{x+y}{x-y}\right) = a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

We have, $\sec\left(\frac{x+y}{x-y}\right) = a \Rightarrow \frac{x+y}{x-y} = \sec^{-1} a$

On differentiating both the sides, we get

$$\frac{\left(1+\frac{dy}{dx}\right)(x-y) - (x+y)\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow (x - y - x - y) + (x - y + x + y)\frac{dy}{dx} = 0$$

$$\Rightarrow -2y + 2x\frac{dy}{dx} = 0 \Rightarrow 2x\frac{dy}{dx} = 2y \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

27. Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

The equation of the curve is $y = 2x^3 - 3$

Differentiating with respect to x, we get $\frac{dy}{dx} = 6x^2$

Now, m_1 = (Slope of the tangent at $x = 2$)

$$= \left(\frac{dy}{dx}\right)_{x=2} = 6 \times (2)^2 = 24$$

and, m_2 = (Slope of the tangent at $x = -2$)

$$= \left(\frac{dy}{dx}\right)_{x=-2} = 6(-2)^2 = 24$$

Clearly, $m_1 = m_2$

Thus, the tangents to the given curve at the points where $x = 2$ and $x = -2$ are parallel.

28. Solve: $\frac{dy}{dx} = 1 - x + y - xy$

We have, $\frac{dy}{dx} = 1 - x + y - xy$

$$\Rightarrow \frac{dy}{dx} = (1-x) + y(1-x) \Rightarrow \frac{dy}{1+y} = (1-x)dx$$

$$\Rightarrow \int \frac{dy}{1+y} = \int (1-x)dx$$

$$\Rightarrow \log(1+y) = x - \frac{x^2}{2} + C$$

SECTION – IV

Questions 29 to 35 carry 3 marks each.

29. Let the function $f: R^+ \to [-9, \infty)$ given by f(x) = 5x² + 6x – 9. Prove that f is bijective.

We have a mapping $f: R^+ \to (-9, \infty)$

given by $f(x) = 5x^2 + 6x - 9$

For one-one Let $x_1, x_2 \in R^+$ be any arbitrary elements, such that $f(x_1) = f(x_2)$

$\Rightarrow 5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$

$\Rightarrow 5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow 5(x_1 + x_2)(x_1 - x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)[5(x_1 + x_2) + 6] = 0$

$\Rightarrow x_1 - x_2 = 0$ or $5x_1 + 5x_2 + 6 = 0$

$\Rightarrow x_1 - x_2 = 0$ [$\because 5x_1 + 5x_2 + 6 \neq 0$ as $x_1, x_2 \in R^+$]

$\Rightarrow x_1 = x_2$

So, f is one-one.

For onto Let $y \in (-9, \infty)$ be any arbitrary element and $y = f(x)$.

Then, $y = 5x^2 + 6x - 9 \Rightarrow 5x^2 + 6x - 9 - y = 0$

$$= x = \frac{-6 \pm \sqrt{36 + 4 \times 5(9+y)}}{10} \left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\Rightarrow x = \frac{-6 \pm \sqrt{216 + 20y}}{10} \Rightarrow x = \frac{-6 \pm 2\sqrt{54 + 5y}}{10}$$

$$\Rightarrow x = \frac{-3 + \sqrt{54 + 5y}}{5} \text{ or } x = \frac{-3 - \sqrt{54 + 5y}}{5}$$

$\because x \in R^+$, therefore $x \neq \frac{-3 - \sqrt{54 + 5y}}{5}$

Now, $x = \frac{-3 + \sqrt{54 + 5y}}{5} \in R^+$ for each $y \in (-9, \infty)$

$\because -9 < y < \infty$

$\Rightarrow -45 < 5y < \infty$ [multiply each term by 5]

$\Rightarrow 54 - 45 < +5y < \infty$ [add 54 in each term]

$\Rightarrow 9 < 54 + 5y < \infty$

$\Rightarrow 3 < \sqrt{54 + 5y} < \infty$ [taking square root]

$\Rightarrow -3 + 3 < -3 + \sqrt{54 + 5y} < \infty$ [subtract –3 in each term]

$\Rightarrow -3 + \sqrt{54 + 5y} > 0$

Thus, for each $y \in (-9, \infty)$, there exists

$$x = \frac{-3 + \sqrt{54 - 5y}}{5} \in R^+ \text{ such that } f(x) = y$$

So, f is onto.

30. Evaluate: $\int_3^4 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} dx$

Let $I = \int_3^4 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} dx$ ... (i)

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ in Eq. (i), we get

$$I = \int_3^4 \frac{\sqrt{3+4-x}}{\sqrt{3+4-x} + \sqrt{7-(3+4-x)}} dx = \int_3^4 \frac{\sqrt{7-x}}{\sqrt{7-x} + \sqrt{x}} \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_3^4 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{7-x}} dx + \int_3^4 \frac{\sqrt{7-x}}{\sqrt{7-x} + \sqrt{x}} dx = \int_3^4 \frac{\sqrt{x} + \sqrt{7-x}}{\sqrt{x} + \sqrt{7-x}} dx = \int_3^4 dx = [x]_3^4$$

$$\Rightarrow 2I = [4-3] = 1$$

$$\therefore I = \frac{1}{2}$$

31. Find the solution of differential equation $x^2dy + y(x + y)dx = 0$, if x = 1 and y = 1.

Given differential equation $x^2dy + y(x+y)dx = 0$ can be written as $x^2dy + (xy + y^2)dx = 0$

$\Rightarrow x^2dy = -(xy + y^2)dx$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{yx + y^2}{x^2}\right)$

$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$ ...(i)

which is a homogeneous, as $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i), we get

$$v + x\frac{dv}{dx} = -v - v^2$$

$\Rightarrow x\frac{dv}{dx} = -2v - v^2 \Rightarrow \frac{1}{2v + v^2}dv = -\frac{1}{x}dx$

$\Rightarrow \frac{1}{v(2+v)}dv = -\frac{1}{x}dx \Rightarrow \frac{2}{2v(2+v)}dv = -\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right)dv = -\frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\int\frac{1}{v}dv - \frac{1}{2}\int\frac{1}{v+2}dv = -\int\frac{1}{x}dx + \log C$

$\Rightarrow \frac{1}{2}\log|v| - \frac{1}{2}\log|v+2| = -|\log x| + \log C$

$\Rightarrow \frac{1}{2}\log\left|\frac{v}{v+2}\right| = \log\left|\frac{C}{x}\right|$

$\Rightarrow \log\left|\frac{v}{v+2}\right| = 2\log\left|\frac{C}{x}\right| \Rightarrow \frac{v}{v+2} = \left(\frac{C}{x}\right)^2$

$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2} = \left(\frac{C}{x}\right)^2$ $\left[\text{put } v = \frac{y}{x}\right]$ $\Rightarrow \frac{y}{y+2x} = \left(\frac{C}{x}\right)^2$

It is given that $y = 1$, when $x = 1$.

$\therefore \frac{1}{1+2} = \left(\frac{C}{1}\right)^2 \Rightarrow \frac{1}{3} = C^2$

Hence, the particular solution of the given differential equation is

$$\frac{y}{y+2x} = \frac{\frac{1}{3}}{x^2} \Rightarrow \frac{y}{y+2x} = \frac{1}{3x^2}$$

$\therefore 3x^2y = y + 2x$

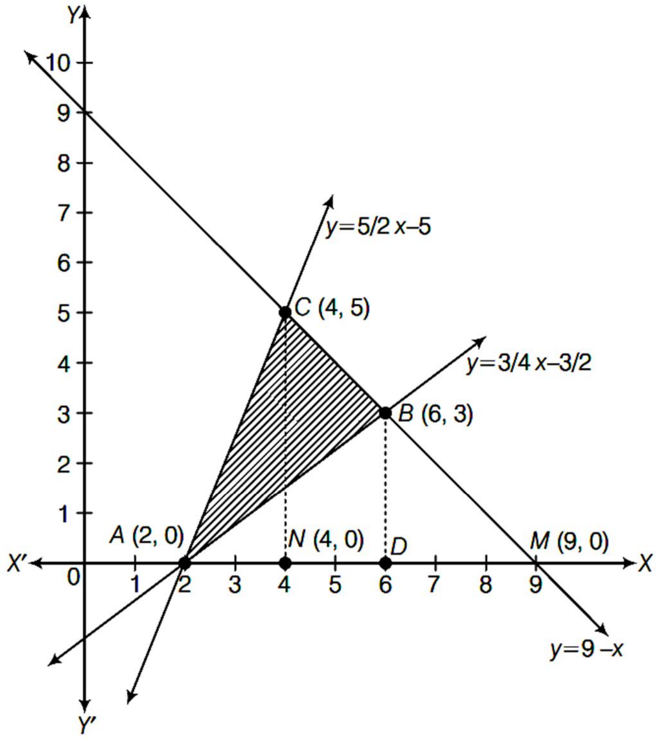
32. Find the area of region bounded by lines $y = \frac{5}{2}x - 5$, $x + y - 9 =- 0$ and $y = \frac{3}{4}x - \frac{3}{2}$.

Given lines are $y = \frac{5}{2}x - 5$ ...(i)

$x + y = 9$ ...(ii)

and $y = \frac{3}{4}x - \frac{3}{2}$ ...(iii)

For finding the points of intersection, we solve in pairs.
On solving Eqs. (i) and (ii), we get
Coordinates of $C = (4, 5)$
On solving Eqs. (ii) and (iii), we get
Coordinates of $B = (6, 3)$
On solving Eqs. (i) and (iii), we get
Coordinates of $A = (2, 0)$



$\therefore$ Required area = Area of ΔANC + Area of quadrilateral $NCBD$ – Area of ΔABD

$= \int_2^4 (\text{line } AC)dx + \int_4^6 (\text{line } BC)dx - \int_2^6 (\text{line } AB)dx$

$= \int_2^4 \left(\frac{5}{2}x - 5\right)dx + \int_4^6 (9 - x)dx - \int_2^6 \left(\frac{3}{4}x - \frac{3}{2}\right)dx$

$= \left[\frac{5}{2}\cdot\frac{x^2}{2} - 5x\right]_2^4 + \left[9x - \frac{x^2}{2}\right]_4^6 - \left[\frac{3}{4}\cdot\frac{x^2}{2} - \frac{3}{2}x\right]_2^6$

$= \frac{5}{4}[x^2]_2^4 - 5[x]_2^4 + 9[x]_4^6 - \frac{1}{2}[x^2]_4^6 - \frac{3}{8}[x^2]_2^6 + \frac{3}{2}[x]_2^6$ $= \frac{5}{4}(16-4) - 5(4-2) + 9(6-4) - \frac{1}{2}[36-16] - \frac{3}{8}(36-4) + \frac{3}{2}(6-2)$

$= \frac{5}{4}(12) - 5(2) + 9(2) - \frac{1}{2}(20) - \frac{3}{8}(32) + \frac{3}{2}(4)$

$= 15 - 10 + 18 - 10 - 12 + 6 = |39 - 32| = 7$ sq units

33. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Let ABC be a triangle inscribed in a given circle with centre O and radius r.

The area of the triangle will be maximum if its vertex A opposite to the base BC is at a maximum distance from the base BC.

This is possible only when A lies on the diameter perpendicular to BC. Thus, $AD \perp BC$. So, ΔABC must be an isoscele triangle. Let $OD = x$.

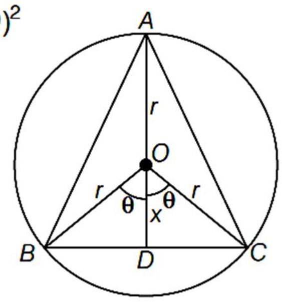
On applying Pythagoras theorem in right angled ΔODB, we get

$$(OB)^2 = (OD)^2 + (BD)^2$$

$$\Rightarrow \quad r^2 = x^2 + (BD)^2$$

$$\Rightarrow \quad BD = \sqrt{r^2 - x^2}$$

$$\therefore \quad BC = 2\,BD = 2\sqrt{r^2 - x^2}$$



Also, $AD = AO + OD = r + x$.

Let A denote the area of ΔABC.

Then, $A = \frac{1}{2}(BC \times AD) \Rightarrow A = \frac{1}{2} \times 2\sqrt{r^2 - x^2} \times (r + x)$

$$\Rightarrow \quad A = (r + x)\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \sqrt{r^2 - x^2} - \frac{x(r + x)}{\sqrt{r^2 - x^2}} \quad \text{[differentiating w.r.t. } x]$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow \quad (r - 2x)(r + x) = 0 \qquad [\because r + x \neq 0]$$

$$\Rightarrow \quad r - 2x = 0 \Rightarrow x = \frac{r}{2}$$

Now, $\frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{(-r - 4x)}{\sqrt{r^2 - x^2}} + \frac{(r^2 - rx - 2x^2)x}{(r^2 - x^2)^{3/2}}$$

[differentiating both sides w.r.t. x]

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x = \frac{r}{2}} = -2\sqrt{3} < 0$$

Thus, A is maximum when $x = \frac{r}{2}$.

$$\therefore \quad BD = \sqrt{r^2 - x^2} = \sqrt{r^2 - \frac{r^2}{4}} \Rightarrow BD = \frac{\sqrt{3}r}{2}$$

In right angled ΔODB, $\tan\theta = \frac{BD}{OD}$

$$\Rightarrow \tan\theta = \frac{\frac{\sqrt{3}r}{2}}{\frac{r}{2}} = \sqrt{3} \Rightarrow \theta = 60°$$

$$\therefore \quad \angle BAC = \theta = 60°$$

But $AB = AC$. Therefore, $\angle B = \angle C = 60°$.

Thus, we obtain $\angle A = \angle B = \angle C = 60°$.

Hence, A is maximum when ABC is an equilateral triangle.

34. If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

Given, $x^y - y^x = a^b$

Let $x^y = u$ and $y^x = v$

Then, $u - v = a^b \Rightarrow \frac{du}{dx} - \frac{dv}{dx} = 0 \quad \ldots(i)$

Now, $u = x^y \Rightarrow \log u = y \log x$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \log x\frac{dy}{dx} \Rightarrow \frac{du}{dx} = y \cdot x^{y-1} + x^y \cdot \log x\frac{dy}{dx}$$

and $v = y^x \Rightarrow \log v = x \log y$

$$\Rightarrow \quad \frac{1}{v} \cdot \frac{dv}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y \Rightarrow \frac{dv}{dx} = xy^{x-1}\frac{dy}{dx} + y^x \log y$$

Now, Eq. (i) becomes,

$$y \cdot x^{y-1} + x^y \cdot \log x\frac{dy}{dx} - xy^{x-1}\frac{dy}{dx} - y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx}(x^y \log x - xy^{x-1}) = y^x \log y - y \cdot x^{y-1} \Rightarrow \frac{dy}{dx} = \frac{y^x \cdot \log y - y \cdot x^{y-1}}{x^y \cdot \log x - x \cdot y^{x-1}}$$

35. Find whether the following function is differentiable at x = 1 and x = 2 or not.

$$f(x) = \begin{cases} x & , \; x < 1 \\ 2 - x & , 1 \le x \le 2 \\ -2 + 3x - x^2 , & x > 2 \end{cases}$$

Given, $f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$

Differentiability at $x = 1$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h) - [2-(1)]}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{2 - (1+h) - (2-1)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = -1$$

$\because$ LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 1$.

Differentiability at $x = 2$

$$\text{LHD} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{2 - (2-h) - (2-2)}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{-2 + 3(2+h) - (2+h)^2 - (2-2)}{h}$$

$$= \lim_{h \to 0} \frac{-2 + 6 + 3h - (4 + h^2 + 4h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{-h^2 - h}{h} = \lim_{h \to 0} \frac{-h(h+1)}{h} = -(0+1) = -1$$

$\because$ LHD = RHD

So, $f(x)$ is differentiable at $x = 2$.

Hence, $f(x)$ is not differentiable at $x = 1$, but it differentiable at $x = 2$.

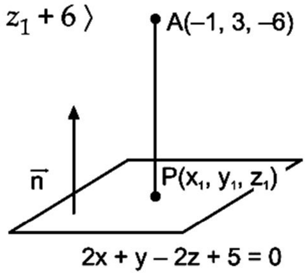
SECTION – V

Questions 36 to 38 carry 5 marks each.

36. Find the foot of the perpendicular drawn from the point (–1, 3, –6) to the plane 2x + y – 2z + 5 = 0. Also, find the equation and length of the perpendicular.

Let the foot of perpendicular be $P(x_1, y_1, z_1)$ dr's of $\overrightarrow{AP}$ are $\langle x_1 + 1, y_1 - 3, z_1 + 6 \rangle$

Direction ratios normal to plane (1) are $\langle 2, 1, -2 \rangle$



Since $AP \parallel \vec{n}$

$$\Rightarrow \frac{x_1 + 1}{2} = \frac{y_1 - 3}{1} = \frac{z_1 + 6}{-2} = \lambda$$

$$\Rightarrow x_1 = 2\lambda - 1, y_1 = \lambda + 3, z_1 = -2\lambda - 6$$

Putting in equation of plane :

$$2x_1 + y_1 - 2z_1 + 5 = 0$$

$$2(2\lambda - 1) + \lambda + 3 - 2(-2\lambda - 6) + 5 = 0 \Rightarrow 9\lambda + 18 = 0 \Rightarrow \lambda = -2$$

$\therefore x_1 = -5, y_1 = 1, z_1 = -2$

Hence foot of perpendicular is P(– 5, 1, – 1).

$$\text{Length of AP} = \sqrt{(-5+1)^2 + (1-3)^2 + (-2+6)^2} = 6 \text{ units}$$

Since $AP \parallel \vec{n}$

$$\therefore \text{Equation of AP is } \frac{x-(-1)}{2} = \frac{y-3}{1} = \frac{z+6}{-2} \Rightarrow \frac{x+1}{2} = \frac{y-3}{1} = \frac{z+6}{-2}$$

37. Solve the LPP graphically minimize, Z = 30x – 30y + 1800 subject to constraints x + y ≤ 30, x ≤ 15, y ≤ 20, x + y ≥ 15; x, y ≥ 0.

Minimise $Z = 30x - 30y + 1800$

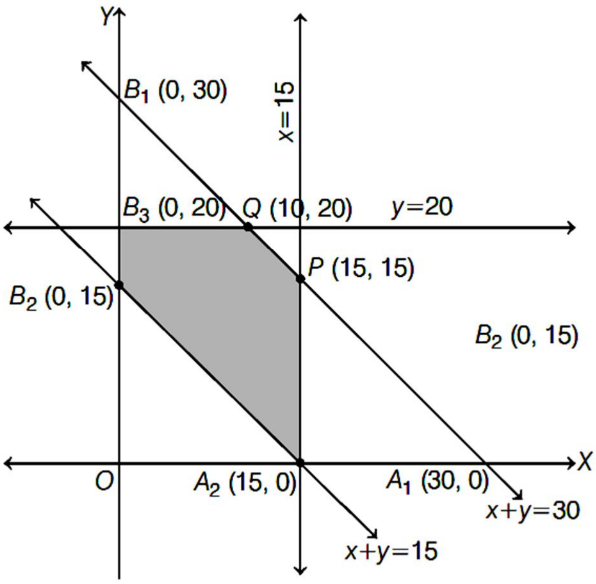
Subject to $x + y \le 30$

$x \le 15$

$y \le 20$

$x + y \ge 15$

and, $x \ge 0, y \ge 0$



To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in given figure The coordinates of the corner points of the feasible region $A_2 PQ B_3 B_2$ are $A_2(15, 0)$, $P(15, 15)$, $Q(10, 20)$, $B_3(0, 20)$ and $B_2(0, 15)$. These points have been obtained by solving the corresponding intersecting lines simultaneously.

The values of the objective function at the points of the feasible region are given in the following table.

Point (x, y)	Value of the objective function $Z = 30x - 30y + 1800$
$A_2(15, 0)$	$Z = 30 \times 15 - 30 \times 0 + 1800 = 2250$
$B(15, 15)$	$Z = 30 \times 15 - 30 \times 15 + 1800 = 1800$
$Q(10, 20)$	$Z = 30 \times 10 - 30 \times 20 + 1800 = 1500$
$B_3(0, 20)$	$Z = 30 \times 0 - 30 \times 20 + 1800 = 1200$
$B_2(0, 15)$	$Z = 30 \times 0 - 30 \times 15 + 1800 = 1350$

Clearly, Z is minimum at $x = 0$, $y = 20$ and the minimum value of Z is 1200.

38. If $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, then find A^{-1}. Using A^{-1}, solve the given system of linear equations:

x + 3y + 4z = 8, 2x + y + 2z = 5 and 5x + y + z = 7.

We have, $x + 3y + 4z = 8,$

$2x + y + 2z = 5$

and $5x + y + z = 7$

The above system of simultaneous linear equations can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

or, $AX = B$, where $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 1(1-2) - 3(2-10) + 4(2-5)$

$= -1 + 24 - 12 = 11 \neq 0$

So, A^{-1} exists.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$C_{11} = -1, C_{12} = 8, C_{13} = -3, C_{21} = 1, C_{22} = -19, C_{23} = 14,$
$C_{31} = 2, C_{32} = 6$ and $C_{33} = -5$

$$\therefore \text{adj } A = \begin{bmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Thus, the solution of the system of equations is given by

$$X = A^{-1}B = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -8 & +5 & +14 \\ 64 & -95 & +42 \\ -24 & +70 & -35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x = 1, y = 1 \text{ and } z = 1$$